

# The Sign of Fourier Coefficients of Half-Integral Weight Cusp Forms

Thomas A. Hulse, E. Mehmet Kiral, Chan Jeong Kuan, Li-Mei Lim

*Department of Mathematics, Brown University, 151 Thayer St.*

*Providence, RI 02912, USA*

*{tahulse,e.mehmet,ck9,llim}@math.brown.edu*

From a result of Waldspurger [6], it is known that the normalized Fourier coefficients  $a(m)$  of a half-integral weight holomorphic cusp eigenform  $\mathfrak{f}$  are, up to a finite set of factors, one of  $\pm\sqrt{L(\frac{1}{2}, f, \chi_m)}$  when  $m$  is square-free and  $f$  is the integral weight cusp form related to  $\mathfrak{f}$  by the Shimura correspondence [8]. In this paper we address a question posed by Kohnen: which square root is  $a(m)$ ? In particular, if we look at the set of  $a(m)$  with  $m$  square-free, do these Fourier coefficients change sign infinitely often? By partially analytically continuing a related Dirichlet series, we are able to show that this is so.

## 1. Introduction

Let  $k$  be an odd integer and  $\mathfrak{f} \in S_{\frac{k}{2}}(\Gamma_0(4))$ ; that is, a cusp form of half-integral weight  $k/2$  and level 4 as described by Shimura [8]. We will discuss the automorphic properties of  $\mathfrak{f}$  in more detail in Section 2. Let the Fourier expansion of  $\mathfrak{f}$  at  $\infty$  be

$$\mathfrak{f}(z) = \sum_{m=1}^{\infty} a(m) m^{\frac{k}{4}-\frac{1}{2}} e(mz). \quad (1.1)$$

The Shimura correspondence provides a holomorphic modular form  $f$  of weight  $k-1$  such that if  $\mathfrak{f}$  is an eigenform of the Hecke operator  $T_{\frac{k}{2}}(p^2)$ , then  $f$  is an eigenform of  $T_{k-1}(p)$  with the same eigenvalue. It was proven in [7] that the level of  $f$  is 2. For the definitions of the Hecke operators in the half-integral weight case, see [8]. Waldspurger proved in [9] that for any square-free  $t$ ,

$$a(t)^2 = c \cdot L\left(\frac{1}{2}, f, \chi_t\right) \quad (1.2)$$

where

$$\chi_t(n) = \left( \frac{(-1)^{\frac{k}{2}-\frac{1}{2}} t}{n} \right) \quad (1.3)$$

is the unique real primitive character modulo  $t$ . The constant  $c$  is dependent only on  $\mathfrak{f}$ . Later this result was made explicit by Kohnen and Zagier in the case of  $\mathfrak{f} \in S_{k+\frac{1}{2}}^+(\Gamma_0(4))$ , see [6] for the definition of the space  $S_{k+\frac{1}{2}}^+(\Gamma_0(4))$ . For a fundamental discriminant  $D$  satisfying

$$(-1)^{\frac{k}{2}-\frac{1}{2}} D > 0, \quad (1.4)$$

they proved that

$$\frac{a(|D|)^2}{\langle \mathfrak{f}, \mathfrak{f} \rangle} = \frac{(\frac{k-1}{2} - 1)!}{\pi^{(k-1)/2}} \frac{L(\frac{1}{2}, f, \chi_D)}{\langle f, f \rangle}. \quad (1.5)$$

Here  $\langle \mathfrak{f}, \mathfrak{f} \rangle$  and  $\langle f, f \rangle$  are the normalized Petersson inner products, and

$$\chi_D(n) = \left( \frac{D}{n} \right) \quad (1.6)$$

is the Kronecker symbol.

The relationship (1.2) prompts the questions posed by Kohnen: which square root of  $L(\frac{1}{2}, f, \chi_t)$  is  $a(t)$  proportional to, and how often? In [5], Kohnen in fact proves that for any half-integral weight cusp form  $\mathfrak{f} \in S_{k+\frac{1}{2}}(N, \chi)$ , not necessarily an eigenform, the sequence of Fourier coefficients  $a(tn^2)$  for a fixed  $t$  square-free has infinitely many sign changes. A natural next question one may ask is whether all the Fourier coefficients  $a(t)$  with  $t$  running over square-free integers change sign infinitely often. In the following theorem we prove that this is indeed the case for eigenforms.

**Theorem 1.1.** *Given  $\mathfrak{f} \in S_{\frac{k}{2}}(\Gamma_0(4))$ , an eigenform of all Hecke operators  $T_{\frac{k}{2}}(p^2)$  for  $p$  prime, where  $k$  is an odd integer, with Fourier expansion*

$$\mathfrak{f}(z) = \sum_{m=1}^{\infty} a(m) m^{\frac{k}{4} - \frac{1}{2}} e(mz); \quad (1.7)$$

*the Fourier coefficients  $a(t)$ , with  $t$  running over square-free integers, change sign infinitely often.*

Inspired by the methods in [1] and [2], we will prove Theorem 1.1 by analytically continuing the Dirichlet series

$$\sum_{\substack{t \geq 1 \\ t \text{ square-free}}} \frac{a(t)}{t^s} \quad (1.8)$$

to  $\Re(s) > 3/4$  by exploiting the analytic continuations of a family of Mellin transforms related to  $\mathfrak{f}$ ; we then prove our claim by contradiction.

## 2. Automorphic Properties

Before we proceed, we will review the automorphic properties of half-integral weight cusp forms. Let  $\mathfrak{f}$  be as above. Given  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ , we have

$$j(\gamma, z) := \varepsilon_d^{-1} \left( \frac{c}{d} \right) (cz + d)^{\frac{1}{2}} = \theta(\gamma(z))/\theta(z) \quad (2.1)$$

where  $\left( \frac{c}{d} \right)$  is Shimura's extension of the Jacobi symbol as in [8]. Setting  $\xi := (\gamma, j(\gamma, z))$ ,  $\mathfrak{f}$  satisfies

$$\mathfrak{f}|_{[\xi]_k}(z) := j(\gamma, z)^{-k} \mathfrak{f}(z) = \varepsilon_d^k \left( \frac{c}{d} \right) (cz + d)^{-\frac{k}{2}} \mathfrak{f} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} z \right) = \mathfrak{f}(z). \quad (2.2)$$

Here  $\varepsilon_d$  is the sign of the Gaussian sum  $\sum_{n=1}^d e(\frac{n^2}{d})$ :

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4} \\ i & \text{if } d \equiv 3 \pmod{4}. \end{cases} \quad (2.3)$$

Furthermore, we fix the following expressions for  $\mathfrak{f}_{\frac{1}{2}}$  and  $\mathfrak{f}_0$ , which are evaluations of  $\mathfrak{f}$  at the respective cusps  $\frac{1}{2}$  and 0, as

$$\mathfrak{f}_{\frac{1}{2}}(z) := \mathfrak{f} \left| \left[ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \right]_k \right. (z) = (-2z + 1)^{-\frac{k}{2}} \mathfrak{f} \left( \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} z \right) \quad (2.4)$$

$$\mathfrak{f}_0(z) := \mathfrak{f} \left| \left[ \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \right]_k \right. (z) = (-2iz)^{-\frac{k}{2}} \mathfrak{f} \left( \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} z \right). \quad (2.5)$$

Also note that  $\mathfrak{f} \left( \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} z \right) = \mathfrak{f}(z)$  for all  $r \in \mathbb{R}^*$ .

### 3. Arguing by Contradiction

Our proof by contradiction proceeds as follows. Take the Dirichlet series

$$M(s) = \sum_{\substack{t \geq 1 \\ t \text{ square-free}}} \frac{a(t)}{t^s} \quad (3.1)$$

as described in (1.8) and assume that  $a(t)$  changes sign finitely many times. Assume for a contradiction that  $a(t) \geq 0$  for  $t > T$  where  $T$  is sufficiently large. Throughout this section, we let  $t$  denote a square-free positive integer.

Suppose that  $M(s)$  analytically continues to  $\Re(s) > \frac{3}{4}$  with polynomial growth in  $\Im(s)$ , as this work will demonstrate. Using a well-known inverse Mellin transform, we get

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} M(s) \Gamma(s) x^s ds = \sum_t a(t) e^{-t/x}. \quad (3.2)$$

The integral on the left-hand side above is  $O(x^{3/4+\varepsilon})$  for any  $\varepsilon > 0$ , as we can move the line of integration to  $\Re(s) = 3/4 + \varepsilon$ . On this vertical line, the gamma function decreases exponentially, whereas the analytic continuation of  $M(s)$  only has polynomial growth, as will be shown below in Proposition 4.4. Since the integrand has no poles for  $\Re(s) > 3/4$ , we don't pick up any residues in moving the line of integration. Thus we arrive at the inequality

$$\sum_t a(t) e^{-t/x} \ll x^{3/4+\varepsilon}. \quad (3.3)$$

The completed Eisenstein series for level 4 is

$$E^*(z, s) = 2^{2s-1} \zeta^*(2s) E(z, s) = 2^{2s-1} \zeta^*(2s) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \Im(\gamma z)^s, \quad (3.4)$$

where  $\zeta^*(s)$  is given by

$$\zeta^*(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s). \quad (3.5)$$

The completed Eisenstein series is holomorphic except for simple poles at  $s = 0, 1$ , with residues  $-\frac{1}{4}$  and  $\frac{1}{4}$  respectively. We have the identity

$$\iint_{\Gamma_0(4) \backslash \mathfrak{h}} |\mathfrak{f}(z)|^2 E^*(z, s) y^{k/2} \frac{dx dy}{y^2} = \Gamma\left(s + \frac{k}{2} - 1\right) 2^{1-k} \pi^{-(s+\frac{k}{2}-1)} \zeta^*(2s) L^{(2)}(\mathfrak{f}, s), \quad (3.6)$$

where

$$L^{(2)}(\mathfrak{f}, s) = \sum_{m=1}^{\infty} \frac{a(m)^2}{m^s} \quad (3.7)$$

which follows after a Rankin-Selberg unfolding. This implies that  $L^{(2)}(\mathfrak{f}, s)$  has a pole at  $s = 1$  with a non-zero residue. In fact, due to the integral representation above, the Rankin-Selberg convolution  $L$ -series extends to a meromorphic function with the only pole at  $s = 1$  when  $\Re(s) \geq \frac{1}{2}$ .

Considering the inverse Mellin transform

$$I = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L^{(2)}(\mathfrak{f}, s) \Gamma(s) x^s ds = \sum_m a(m)^2 e^{-m/x}, \quad (3.8)$$

and shifting the line of integration to  $\Re(s) = \frac{1}{2}$  past the pole at  $s = 1$ , we get

$$I = (\text{Res}_{s=1} L^{(2)}(\mathfrak{f}, s)) x + \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L^{(2)}(\mathfrak{f}, s) \Gamma(s) x^s ds, \quad (3.9)$$

which implies, as the contribution from the integral above is  $O(x^{\frac{1}{2}})$ , that

$$x \ll \sum_m a(m)^2 e^{-m/x}. \quad (3.10)$$

In the above sum, the square-free integers play a nontrivial role. Indeed, using Lemma 4.1 we will be able to conclude that for any  $\varepsilon > 0$  and  $\Re(s) = \sigma$

$$\begin{aligned} L^{(2)}(\mathfrak{f}, s) &= \sum_{m=1}^{\infty} \frac{a(m)^2}{m^s} = \sum_t \sum_{n=1}^{\infty} \frac{a(tn^2)^2}{(tn^2)^s} \\ &\ll \sum_t \sum_{n=1}^{\infty} \frac{a(t)^2}{t^\sigma n^{2\sigma-2\varepsilon}} = \zeta(2\sigma-2\varepsilon) M^{(2)}(\sigma) \end{aligned} \quad (3.11)$$

where

$$M^{(2)}(s) = \sum_t \frac{a(t)^2}{t^s}. \quad (3.12)$$

Therefore, as we move  $\sigma$  towards  $1^+$  along the real line, since  $L^{(2)}(\mathfrak{f}, s)$  has a pole at 1 and  $\zeta(2\sigma-2\varepsilon) \ll 1$  for  $\sigma \geq 1$ , it follows from (3.11) that  $\sum_t a(t)^2 / t^\sigma$  tends to infinity. Therefore the function  $M^{(2)}(s)$ , which is an analytic function on the region  $\Re(s) > 1$ , has a singularity as  $s$  tends to 1. Put

$$A(x) = \sum_{T < t \leq x} a(t)^2. \quad (3.13)$$

We claim that the singularity of  $M^{(2)}(s)$  forces the partial sums,  $A(x)$ , to not be of slow growth. Indeed, assume that for some  $c < 1$ ,

$$A(x) = O(x^c), \quad (3.14)$$

from the partial summation formula we get, for  $\Re(s) > 1$

$$\sum_{t>T} \frac{a(t)^2}{t^s} = s \int_T^\infty \frac{A(u)}{u^{s+1}} du. \quad (3.15)$$

Then, since we assumed  $A(x) \ll x^c$ , the right-hand side above is an analytic function on the right half plane  $\Re(s) > c$ , but the left-hand side  $M^{(2)}(s)$  has a singularity at  $s = 1$ , giving our contradiction. Thus, for every  $c$  with  $0 < c < 1$ , every constant  $\alpha > 0$ , and every  $x$ , there is an  $x_0 > x$  such that

$$A(x_0) \geq \alpha x_0^c. \quad (3.16)$$

Now from (3.3) we have a constant  $\beta > 0$  such that,

$$\beta x^{3/4+\varepsilon} \geq e \left| \sum a(t) e^{-t/x} \right| \geq e \left| \sum_{t>T} a(t) e^{-t/x} \right| - e \left| \sum_{t \leq T} a(t) e^{-t/x} \right|, \quad (3.17)$$

so using our assumption on the eventual non-negativity of  $a(t)$ , we have that

$$\beta x^{3/4+\varepsilon} + O(1) \geq e \sum_{t>T} a(t) e^{-t/x} \geq \sum_{T < t < x} a(t), \quad (3.18)$$

where  $\varepsilon > 0$  is arbitrarily small, and  $\beta$  depends on  $\varepsilon$ . Thus increasing  $\beta$  to accommodate the constant term, we get

$$\beta x^{3/4+\varepsilon} \geq \sum_{T < t < x} a(t). \quad (3.19)$$

Let  $\lambda n^\theta$  be an upper bound on the individual coefficient  $a(n)$  of the half-integral weight modular form  $\mathfrak{f}$ ; according to [3] one may take  $\theta = 3/14$ . Now apply (3.16) and get that for some  $x_0$  as above, which we may choose to be arbitrarily large,

$$\lambda \beta x_0^{3/4+\varepsilon} \geq \lambda \sum_{T < t \leq x_0} a(t) \geq \sum_{T < t \leq x_0} \frac{a(t)^2}{t^\theta} \geq x_0^{-\theta} \sum_{T < t \leq x_0} a(t)^2 \geq \alpha x_0^{-\theta+c}, \quad (3.20)$$

again by our non-negativity assumption. This implies,

$$x_0^{c-\varepsilon-27/28} \leq \frac{\lambda \beta}{\alpha}. \quad (3.21)$$

By choosing  $c$  and  $\varepsilon$  appropriately we may make the exponent on the left hand side greater than 0, giving our contradiction. Therefore, the assumption that all Fourier coefficients  $a(t)$  change sign finitely many times for square-free  $t$  must be false. Thus, in order to show that the Fourier coefficients  $a(t)$  change sign infinitely often for square-free  $t$ , we need only show that  $M(s)$  can be analytically continued up to the line  $\Re(s) = 3/4$ , and grows slowly on vertical strips. The remainder of this paper is devoted to proving this.

#### 4. Analytic Continuation

We now proceed to obtain an analytic continuation of the Dirichlet series (1.8) to the region  $\Re(s) > 3/4$ . First, note that

$$\sum_{\substack{t \geq 1 \\ t \text{ square-free}}} \frac{a(t)}{t^s} = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} \sum_{r^2 | m} \mu(r) = \sum_{r=1}^{\infty} \mu(r) D_r(s), \quad (4.1)$$

where

$$D_r(s) = \sum_{\substack{m=1 \\ m \equiv 0 \pmod{r^2}}}^{\infty} \frac{a(m)}{m^s}. \quad (4.2)$$

Lemma 4.2 shows that the series  $D_r(s)$  converges for  $\Re(s) > 1$ , and in fact  $D_r(1 + \varepsilon + it) \ll_{\varepsilon} 1/r^2$ . With this fact, we easily see that our Dirichlet series over the square-free integers converges on the half-plane  $\Re(s) > 1$ . We now further examine the series  $D_r(s)$ :

$$\begin{aligned} D_r(s) &= \sum_{\substack{m=1 \\ m \equiv 0 \pmod{r^2}}}^{\infty} \frac{a(m)}{m^s} = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} \left( \frac{1}{r^2} \sum_{u \pmod{r^2}} e\left(\frac{mu}{r^2}\right) \right) \\ &= \frac{1}{r^2} \sum_{u \pmod{r^2}} \sum_{m=1}^{\infty} \frac{a(m) e(\frac{mu}{r^2})}{m^s}. \end{aligned} \quad (4.3)$$

The innermost Dirichlet series can be expressed in terms of an additively twisted Mellin integral of  $\mathfrak{f}$ . For rational  $q$ , denote

$$\Lambda(\mathfrak{f}_*, q, s) = \int_0^{\infty} \mathfrak{f}(iy + q) y^{s + (\frac{k}{4} - \frac{1}{2})} \frac{dy}{y}. \quad (4.4)$$

This integral converges for every  $s \in \mathbb{C}$  because  $q \in \mathbb{Q}$  is a cusp and since  $\mathfrak{f}$  is a cusp form,  $\mathfrak{f}(iy + q)$  decreases exponentially as  $y \rightarrow \infty$  and as  $y \rightarrow 0$ . Thus  $\Lambda(\mathfrak{f}, q, s)$  is an entire function of  $s$ . Let  $\mathfrak{f}_*$  denote  $\mathfrak{f}, \mathfrak{f}_{\frac{1}{2}}$ , or  $\mathfrak{f}_0$ . Now expanding  $\mathfrak{f}_*$  in the integral as its Fourier series, with respective coefficients  $a_*(n)$ , we get:

$$\begin{aligned} \int_0^{\infty} \mathfrak{f}_*(iy + q) y^{s + (\frac{k}{4} - \frac{1}{2})} \frac{dy}{y} &= \int_0^{\infty} \sum_{m=1}^{\infty} a_*(m) m^{(\frac{k}{4} - \frac{1}{2})} e(m(iy + q)) y^{s + (\frac{k}{4} - \frac{1}{2})} \frac{dy}{y} \\ &= \sum_{m=1}^{\infty} a_*(m) m^{(\frac{k}{4} - \frac{1}{2})} e(mq) \int_0^{\infty} e^{-2\pi my} y^{s + (\frac{k}{4} - \frac{1}{2})} \frac{dy}{y} \\ &= \frac{\Gamma(s + (\frac{k}{4} - \frac{1}{2}))}{(2\pi)^{s + (\frac{k}{4} - \frac{1}{2})}} \sum_{m=1}^{\infty} \frac{a_*(m) e(mq)}{m^s}. \end{aligned} \quad (4.5)$$

For ease of notation call, from now on,  $s' := s + (\frac{k}{4} - \frac{1}{2})$ . Using the integral representation of our Dirichlet series, which is that of  $L(\mathfrak{f}, s)$  twisted by an additive character, we obtain

$$\begin{aligned}
 D_r(s) &= \sum_{\substack{m=1 \\ m \equiv 0 \pmod{r^2}}}^{\infty} \frac{a(m)}{m^s} \\
 &= \frac{(2\pi)^{s'}}{\Gamma(s')} \frac{1}{r^2} \sum_{u \pmod{r^2}} \Lambda\left(\mathfrak{f}, \frac{u}{r^2}, s\right) \\
 &= \frac{(2\pi)^{s'}}{\Gamma(s')} \frac{1}{r^2} \sum_{\substack{d|r^2 \\ u \pmod{d}}} \sum_{\substack{(u,d)=1}} \Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right), \tag{4.6}
 \end{aligned}$$

where the fraction  $u/d$  is in lowest terms, and by abuse of notation, we continue to call the numerator  $u$ . Equation (4.6) allows us to express  $D_r(s)$  as a finite sum of entire functions, hence  $D_r(s)$  itself is an entire function. Therefore it makes sense to talk about the growth properties of  $D_r(s)$  in  $r$  for any fixed  $s$  on the complex plane. Also note that we only need to estimate  $D_r(s)$  for square-free  $r$ , due to the existence of  $\mu(r)$  in (4.1).

**Lemma 4.1.** *The Fourier coefficients of a half-integral weight cusp eigenform  $\mathfrak{f} \in S_{\frac{k}{2}}(\Gamma_0(4))$ , with  $k \geq 5$  as above satisfy the following bound*

$$a(tn^2) \ll |a(t)|n^\varepsilon \tag{4.7}$$

where  $t, n \in \mathbb{N}$  and  $t$  is square-free, with the implied constant dependent only on  $\mathfrak{f}$  and  $\varepsilon > 0$ .

**Proof.** The Shimura correspondence [8] gives us that for  $t \in \mathbb{N}$  square-free we have that

$$\sum_{n=1}^{\infty} \frac{a(tn^2)}{n^{s-\frac{k}{2}+1}} = a(t) \left( \sum_{m_1=1}^{\infty} \frac{\chi_t(m_1)\mu(m_1)}{m_1^{s-\frac{k}{2}+\frac{3}{2}}} \right) \left( \sum_{m_2=1}^{\infty} \frac{A(m_2)}{m_2^s} \right), \tag{4.8}$$

where  $\chi_t(m_1) = \left(\frac{-1}{m_1}\right)^{\frac{k}{2}-\frac{1}{2}} \left(\frac{t}{m_1}\right)$  and the  $A(m_2)$  are the Fourier coefficients of the weight  $k-1$  cusp form  $f \in S_{k-1}(\Gamma_0(2))$  such that

$$f(z) = \sum_{m_2=1}^{\infty} A(m_2) e^{2\pi i m_2 z} \tag{4.9}$$

where  $f$  is associated to  $\mathfrak{f}$  by the Shimura correspondence. Expanding the right-

hand side of (4.8) we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a(tn^2)}{n^{s-\frac{k}{2}+1}} &= a(t) \sum_{n=1}^{\infty} \sum_{m_1 m_2 = n} \frac{\chi_t(m_1) \mu(m_1)}{m_1^{s-\frac{k}{2}+\frac{3}{2}}} \frac{A(m_2)}{m_2^s} \\
&= a(t) \sum_{n=1}^{\infty} \sum_{m|n} \frac{\chi_t\left(\frac{n}{m}\right) \mu\left(\frac{n}{m}\right)}{\left(\frac{n}{m}\right)^{s-\frac{k}{2}+\frac{3}{2}}} \frac{A(m)}{m^s} \\
&= a(t) \sum_{n=1}^{\infty} \frac{1}{n^{s-\frac{k}{2}+1}} \sum_{m|n} \frac{\chi_t\left(\frac{n}{m}\right) \mu\left(\frac{n}{m}\right)}{n^{\frac{1}{2}}} \frac{A(m)}{m^{\frac{k}{2}-\frac{3}{2}}}. \tag{4.10}
\end{aligned}$$

Comparing coefficients term-by-term we see that for each  $n$

$$a(tn^2) = a(t) \sum_{m|n} \frac{\chi_t\left(\frac{n}{m}\right) \mu\left(\frac{n}{m}\right)}{n^{\frac{1}{2}}} \frac{A(m)}{m^{\frac{k}{2}-\frac{3}{2}}}. \tag{4.11}$$

Since the Ramanujan-Petersson conjecture is known for integral weight cusp forms, we have  $A(m) \ll m^{\frac{(k-1)-1}{2}+\varepsilon}$  with the implied constant dependent on  $f$  (and thus  $\mathfrak{f}$ ) and  $\varepsilon$ . Using this bound and taking absolute values of (4.11) we get

$$a(tn^2) \ll |a(t)| n^{-\frac{1}{2}} \sum_{m|n} m^{\frac{1}{2}+\varepsilon} \ll |a(t)| n^{\varepsilon-\frac{1}{2}} \sigma_{\frac{1}{2}}(n). \tag{4.12}$$

Since  $\sigma_{\frac{1}{2}}(n) \leq d(n)\sqrt{n}$ , where  $d(n)$  is the divisor function, we have  $\sigma_{\frac{1}{2}}(n) \ll n^{1/2+\varepsilon}$  with the implied constant dependent on  $\varepsilon$ . Putting this into (4.12) gives the desired result.  $\square$

**Lemma 4.2.** *Letting  $r \in \mathbb{N}$  and  $\tau \in \mathbb{R}$ ,*

$$D_r(1 + \varepsilon + i\tau) = \sum_{\substack{m=1 \\ m \equiv 0 \pmod{r^2}}}^{\infty} \frac{a(m)}{m^{1+\varepsilon+i\tau}} \ll \frac{1}{r^2} \tag{4.13}$$

where the implied constant depends only on  $\mathfrak{f}$  and  $\varepsilon$ .

**Proof.** In the sum, we write  $m = nr^2$ , and let  $n_0$  be the square-free part of  $n$ . Then by Lemma 4.1  $a(nr^2) \ll |a(n_0)| \left(\frac{nr^2}{n_0}\right)^{\varepsilon}$ , and therefore for  $s = \sigma + i\tau$  with  $\sigma \geq 1$ ,

$$\sum_{n=1}^{\infty} \frac{a(nr^2)}{n^{s+2\varepsilon}} \ll r^{2\varepsilon} \sum_{n=1}^{\infty} \frac{|a(n_0)| (n/n_0)^{\varepsilon}}{n^{\sigma+2\varepsilon}} \leq r^{2\varepsilon} \sum_{n=1}^{\infty} \frac{|a(n_0)|}{n^{\sigma+\varepsilon}}, \tag{4.14}$$

where the implied constant only depends on  $\mathfrak{f}$  and  $\varepsilon$ . Now,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|a(n_0)|}{n^{\sigma+\varepsilon}} &= \sum_{d=1}^{\infty} \sum_{\substack{n \\ \text{square-free}}} \frac{|a(n)|}{(nd^2)^{\sigma+\varepsilon}} \\
&= \zeta(2\sigma + 2\varepsilon) \sum_{\substack{n \\ \text{square-free}}} \frac{|a(n)|}{n^{\sigma+\varepsilon}} \ll \zeta(2\sigma + 2\varepsilon) \sum_{n=1}^{\infty} \frac{|a(n)|}{n^{\sigma+\varepsilon}} = O(1) \tag{4.15}
\end{aligned}$$



where the implied constant again depends on  $\mathfrak{f}$  and  $\varepsilon$ . Therefore, using both (4.14) and (4.15) we have

$$\begin{aligned} D_r(s + \varepsilon) &= \sum_{\substack{m=1 \\ m \equiv 0 \pmod{r^2}}}^{\infty} \frac{a(m)}{m^{s+\varepsilon}} = \sum_{n=1}^{\infty} \frac{a(nr^2)}{(nr^2)^{s+\varepsilon}} \\ &= \frac{1}{r^{2s+2\varepsilon}} \sum_{n=1}^{\infty} \frac{a(nr^2)}{n^{s+\varepsilon}} \ll \frac{1}{r^{2\sigma}}. \end{aligned} \quad (4.16)$$

Letting  $s = 1 + i\tau$ , we have

$$D_r(1 + \varepsilon + i\tau) \ll \frac{1}{r^2}, \quad (4.17)$$

completing our proof.  $\square$

In order to show that

$$M(3/4 + \varepsilon) = \sum \mu(r) D_r(3/4 + \varepsilon) \quad (4.18)$$

converges, we will bound  $D_r(-\varepsilon + it)$  by a polynomial in  $t$  and apply a Phragmén-Lindelöf convexity argument.

The twisted Mellin integrals  $\Lambda(\mathfrak{f}, \frac{u}{d}, s)$  have functional equations. Depending on the class of equivalent cusps that  $\frac{u}{d}$  belongs to, we get slightly different functional equations. They are as follows:

**Lemma 4.3.** *If  $4|d$ ,*

$$\Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right) = d^{1-2s} (-i)^{\frac{k}{2}} \varepsilon_v^k \left(\frac{d}{v}\right) \Lambda\left(\mathfrak{f}, \frac{v}{d}, 1-s\right), \quad (4.19)$$

where  $v$  is chosen so that  $uv \equiv -1 \pmod{d}$ .

*If  $2||d$ , the functional equation has the type,*

$$\Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right) = d^{1-2s} (-i)^{\frac{k}{2}} \varepsilon_v^k \left(\frac{d}{v}\right) \Lambda\left(\mathfrak{f}_{\frac{1}{2}}, \frac{v}{d}, 1-s\right), \quad (4.20)$$

where once again  $v$  is chosen to satisfy  $uv \equiv -1 \pmod{d}$ .

*Finally, if  $2 \nmid d$ , then*

$$\Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right) = (2d)^{1-2s} \varepsilon_d^{-k} \left(\frac{v}{d}\right) \Lambda\left(\mathfrak{f}_0, \frac{v}{d}, 1-s\right). \quad (4.21)$$

where  $v$  be such that  $4uv \equiv -1 \pmod{d}$ .

**Proof.** First note that

$$\int_0^\infty \mathfrak{f}\left(iy + \frac{u}{d}\right) y^{s+(\frac{k}{4}-\frac{1}{2})} \frac{dy}{y} = \int_0^\infty \mathfrak{f}\left(\begin{pmatrix} 1 & u/d \\ 0 & 1 \end{pmatrix} iy\right) y^{s+(\frac{k}{4}-\frac{1}{2})} \frac{dy}{y}. \quad (4.22)$$

We observe that  $u/d$  is equivalent to the cusps  $\infty, 1/2$ , or  $0$  depending on the conditions  $4|d, 2||d$  or  $2 \nmid d$  respectively. We consider the following matrix decompositions in each case. If  $4|d$ ,

$$\begin{pmatrix} 1 & u/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4/d & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} -u & e \\ -d & v \end{pmatrix} \begin{pmatrix} 1 & v/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}, \quad (4.23)$$

and if  $2||d$ ,

$$\begin{pmatrix} 1 & u/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4/d & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 2e - u & e \\ 2v - d & v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & v/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}, \quad (4.24)$$

where  $v$  and  $e$  are chosen to satisfy  $uv - de = -1$ . Finally for  $d$  odd, with  $v$  and  $e$  chosen to satisfy  $4uv - de = -1$ , we consider the following matrix decomposition:

$$\begin{pmatrix} 1 & u/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/d & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} e & u \\ 4v & d \end{pmatrix} \begin{pmatrix} 0 & 1/4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & v/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}. \quad (4.25)$$

Note that in all of the matrix decompositions, the leftmost matrix is in  $\Gamma_0(4)$ .

Recall that we let  $s' = s + (\frac{k}{4} - \frac{1}{2})$ . For  $4|d$  we use (4.23),

$$\begin{aligned} \Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right) &= \int_0^\infty \mathfrak{f}\left(iy + \frac{u}{d}\right) y^{s+(\frac{k}{4}-\frac{1}{2})} \frac{dy}{y} \\ &= \int_0^\infty \mathfrak{f}\left(\begin{pmatrix} 1 & u/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4/d & 0 \\ 0 & d \end{pmatrix} \frac{id^2y}{4}\right) y^{s'} \frac{dy}{y} \\ &= 4^{s'} d^{-2s'} \int_0^\infty \mathfrak{f}\left(\begin{pmatrix} -u & e \\ -d & v \end{pmatrix} \begin{pmatrix} 1 & v/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} iy\right) y^{s'} \frac{dy}{y} \\ &= d^{-2s'} \int_0^\infty \mathfrak{f}\left(\begin{pmatrix} -u & e \\ -d & v \end{pmatrix} \left(iy + \frac{v}{d}\right)\right) y^{-s'} \frac{dy}{y} \\ &= d^{-2s'} \int_0^\infty \mathfrak{f}|_{[\xi]_k} \left(iy + \frac{v}{d}\right) \varepsilon_v^{-k} \left(\frac{-d}{v}\right) \left(-d\left(iy + \frac{v}{d}\right) + v\right)^{\frac{k}{2}} y^{-s'} \frac{dy}{y} \\ &= d^{1-2s} (-i)^{\frac{k}{2}} \varepsilon_v^k \left(\frac{d}{v}\right) \int_0^\infty \mathfrak{f}\left(iy + \frac{v}{d}\right) y^{1-s+(\frac{k}{4}-\frac{1}{2})} \frac{dy}{y}, \end{aligned} \quad (4.26)$$

where  $\xi = (\gamma, j(\gamma, z))$  for  $\gamma = \begin{pmatrix} -u & e \\ -d & v \end{pmatrix}$  and  $\mathfrak{f}|_{[\xi]_k}$  denotes the slash operator on half-integral weight forms as described in (2.2). Thus, for  $4|d$ ,

$$\Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right) = d^{1-2s} (-i)^{\frac{k}{2}} \varepsilon_v^k \left(\frac{d}{v}\right) \Lambda\left(\mathfrak{f}, \frac{v}{d}, 1-s\right). \quad (4.27)$$

If  $2||d$ , once again let  $v$  and  $e$  satisfy  $uv - de = -1$ . By (4.24) and similar reasoning as above, and using (2.2) and (2.4), we deduce:

$$\begin{aligned}
 \Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right) &= \int_0^\infty \mathfrak{f}\left(iy + \frac{u}{d}\right) y^{s+(\frac{k}{4}-\frac{1}{2})} \frac{dy}{y} \\
 &= d^{-2s'} \int_0^\infty \mathfrak{f}\left(\begin{pmatrix} 2e-u & e \\ 2v-d & v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & v/d \\ 0 & 1 \end{pmatrix} iy\right) y^{-s'} \frac{dy}{y} \\
 &= d^{-2s'} \int_0^\infty \mathfrak{f}|_{[\xi]_k} \left[\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\right]_k \left(iy + \frac{v}{d}\right) \varepsilon_v^{-k} \left(\frac{2v-d}{v}\right)^k (-diy)^{\frac{k}{2}} y^{-s'} \frac{dy}{y} \\
 &= d^{1-2s} (-i)^{\frac{k}{2}} \varepsilon_v^k \left(\frac{d}{v}\right) \int_0^\infty \mathfrak{f}_{\frac{1}{2}}\left(iy + \frac{v}{d}\right) y^{1-s+(\frac{k}{4}-\frac{1}{2})} \frac{dy}{y}
 \end{aligned} \tag{4.28}$$

where, this time,  $\gamma = \begin{pmatrix} 2e-u & e \\ 2v-d & v \end{pmatrix}$ , and  $\xi = (\gamma, j(\gamma, z))$ . Thus for  $2||d$ ,

$$\Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right) = d^{1-2s} (-i)^{\frac{k}{2}} \varepsilon_v^k \left(\frac{d}{v}\right) \Lambda\left(\mathfrak{f}_{\frac{1}{2}}, \frac{v}{d}, 1-s\right). \tag{4.29}$$

For  $d$  odd, we choose  $v, e$  such that  $4uv - de = -1$ . So by (4.25), (2.2) and (2.5) we have

$$\begin{aligned}
 \Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right) &= \int_0^\infty \mathfrak{f}\left(iy + \frac{u}{d}\right) y^{s+(\frac{k}{4}-\frac{1}{2})} \frac{dy}{y} \\
 &= (2d)^{-2s'} \int_0^\infty \mathfrak{f}\left(\begin{pmatrix} e & u \\ 4v & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \left(iy + \frac{v}{d}\right)\right) y^{-s'} \frac{dy}{y} \\
 &= (2d)^{-2s'} \int_0^\infty \mathfrak{f}|_{[\xi]_k} \left[\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}\right] \left(iy + \frac{v}{d}\right) \varepsilon_d^{-k} \left(\frac{v}{d}\right) (-2i)^{\frac{k}{2}} (diy)^{\frac{k}{2}} y^{-s'} \frac{dy}{y} \\
 &= (2d)^{1-2s} \varepsilon_d^{-k} \left(\frac{v}{d}\right) \int_0^\infty \mathfrak{f}_0\left(iy + \frac{v}{d}\right) y^{1-s+(\frac{k}{4}-\frac{1}{2})} \frac{dy}{y}
 \end{aligned} \tag{4.30}$$

where  $\xi = (\gamma, j(\gamma, z))$  and  $\gamma = \begin{pmatrix} e & u \\ 4v & d \end{pmatrix}$ . Thus for  $2 \nmid d$ ,

$$\Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right) = (2d)^{1-2s} \varepsilon_d^{-k} \left(\frac{v}{d}\right) \Lambda\left(\mathfrak{f}_0, \frac{v}{d}, 1-s\right), \tag{4.31}$$

which completes our proof.  $\square$

We now apply our functional equations to the double sum

$$\sum_{d|r^2} \sum_{\substack{(u,d)=1 \\ u \bmod d}} \Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right) \tag{4.32}$$

from (4.6) in order to get an asymptotic bound for  $D_r(s)$  at  $\Re(s) < 0$  in terms of  $r$ . We first split the sum into appropriate parts.

$$\begin{aligned}
& \sum_{d|r^2} \sum_{\substack{(u,d)=1 \\ u \bmod d}} \Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right) \\
&= \overbrace{\sum_{\substack{d|r^2 \\ 4|d}} \sum_{\substack{(u,d)=1 \\ u \bmod d}} \Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right)}^{S_\infty} + \overbrace{\sum_{\substack{d|r^2 \\ 2||d}} \sum_{\substack{(u,d)=1 \\ u \bmod d}} \Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right)}^{S_{\frac{1}{2}}} + \overbrace{\sum_{\substack{d|r^2 \\ 2 \nmid d}} \sum_{\substack{(u,d)=1 \\ u \bmod d}} \Lambda\left(\mathfrak{f}, \frac{u}{d}, s\right)}^{S_0}. \quad (4.33)
\end{aligned}$$

In this expression,  $d$  can be assumed cube-free, since  $r$  can be taken to be square-free and  $d$  ranges over  $d|r^2$ .

Now we estimate this sum for  $s$  slightly to the left of the line  $\Re(s) = 0$ . From (4.5) we have

$$\Lambda\left(\mathfrak{f}_*, \frac{v}{d}, 1 - (-\varepsilon) + i\tau\right) = O_{\varepsilon, \mathfrak{f}}\left((1 + |\tau|)^{\frac{k}{4} + \varepsilon} e^{-\frac{\pi}{2}|\tau|}\right), \quad (4.34)$$

where the implied constant is uniform over all  $v$  and  $d$ , but is dependent on  $\varepsilon$  and  $\mathfrak{f}$ . Using this along with Lemma 4.3 and (4.33), we get that, for  $\varepsilon > 0$ ,

$$\begin{aligned}
& \sum_{d|r^2} \sum_{\substack{(u,d)=1 \\ u \bmod d}} \Lambda\left(\mathfrak{f}, \frac{u}{d}, -\varepsilon + i\tau\right) \\
& \ll_{\varepsilon, \mathfrak{f}} (1 + |\tau|)^{\frac{k}{4} + \varepsilon} e^{-\frac{\pi}{2}|\tau|} \sum_{d|r^2} \varphi(d) d^{1 - (-2\varepsilon)} \\
& \ll_{\varepsilon, \mathfrak{f}} (1 + |\tau|)^{\frac{k}{4} + \varepsilon} e^{-\frac{\pi}{2}|\tau|} \sigma_{2+2\varepsilon}(r^2) \\
& \ll_{\varepsilon, \mathfrak{f}} (1 + |\tau|)^{\frac{k}{4} + \varepsilon} e^{-\frac{\pi}{2}|\tau|} r^{4+5\varepsilon}. \quad (4.35)
\end{aligned}$$

Thus using this estimate in (4.6),

$$D_r(-\varepsilon + i\tau) \ll_{\varepsilon, \mathfrak{f}} (1 + |\tau|)^{1+2\varepsilon} r^{2+5\varepsilon}. \quad (4.36)$$

Using this along with Lemma 4.2, a Phragmén-Lindelöf argument tells us that

$$D_r(3/4 + \varepsilon + i\tau) \ll_{\varepsilon, \mathfrak{f}, \tau} 1/r^{1+4\varepsilon} \quad (4.37)$$

which, when put to use in (4.1), provides

$$M(s) = \sum_{\text{square-free}} \frac{a(t)}{t^s} = \sum_{r=1}^{\infty} \mu(r) D_r(s) \ll_{\varepsilon, \mathfrak{f}, \tau} \sum_{r=1}^{\infty} \frac{1}{r^{1+4\varepsilon}} < \infty, \quad (4.38)$$

where  $s = \sigma + i\tau$  with  $\sigma > 3/4 + \varepsilon$ . Therefore, we have proven the following:

**Proposition 4.4.** *The series*

$$M(s) = \sum_t \frac{a(t)}{t^s} \quad (4.39)$$

*converges in the half plane  $\Re(s) > \frac{3}{4}$  and also has only polynomial growth in  $\Im(s)$  in the vertical strips in that region.*

This was the desired pole-free region to prove Theorem 1.1.

## 5. Acknowledgments

We would like to thank Jeff Hoffstein for introducing us to this problem and for suggesting how we might approach it.

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